Sup and Inf

Problem
Let $S \subseteq \mathbb{R}$ be a nonempty bounded set. If $x \in \mathbb{R}$ is such that $x \leq y$ for all $y \in S$, show that $x \leq \inf S$.

Problem
Let $S \subseteq \mathbb{R}$ be a nonempty bounded set. If $P \subseteq S$ is nonempty, show that it is also bounded, and

$$\sup P \leq \sup S,$$
$$\inf P \geq \inf S.$$

Problem
Let $S \subseteq \mathbb{R}$ be a nonempty bounded set. Show that $\alpha = \sup S$ iff

$$(\forall \epsilon > 0)(\exists x \in S)[\alpha - \epsilon < x \leq \alpha].$$

Formulate and prove a similar statement about $\inf S$.

Problem
Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are bounded functions. Show that $f + g$ and fg are both bounded, and

$$\sup(f + g) \leq \sup(f) + \sup(g),$$
$$\inf(f + g) \geq \inf(f) + \inf(g).$$

Furthermore, prove that if f and g are nonnegative,

$$\sup(fg) \leq \sup(f) \cdot \sup(g),$$
$$\inf(fg) \geq \inf(f) \cdot \inf(g).$$

Give an example of f, g such that $\sup(fg) < \sup(f) \cdot \sup(g)$.

Problem
Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded. Show that

$$\sup(-f) = -\inf(f)$$

Problem
Suppose A, B are bounded subsets of $\mathbb{R}$. Define

$$A + B = \{a + b : a \in A, b \in B\},$$
$$AB = \{ab : a \in A, b \in B\}.$$

Show that $A + B$ and AB are bounded, and

$$\sup(A + B) = \sup(A) + \sup(B),$$
$$\sup(AB) = \sup(A) \cdot \sup(B).$$

Explain the difference between this and the product of two functions and why one needs to be an inequality and the other one doesn't.

Limits

Problem
Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bounded nondecreasing function. Show that

$$\lim_{x\to\infty} f(x) = \sup(f),$$
$$\lim_{x\to-\infty} f(x) = \inf(f).$$

Can we say something similar when f is bounded and nonincreasing?

Problem
Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and $a \in \mathbb{R}$ be given. Define $g, h : (0, \infty) \to \mathbb{R}$ such that

$$g(x) = \inf\{f(y) : 0 < |y - a| < x\},$$
$$h(x) = \sup\{f(y) : 0 < |y - a| < x\}.$$

1. Prove that g is nonincreasing and h is nondecreasing.
2. Prove that $\sup(g) \leq \inf(h)$.
3. Prove that $\sup(g) = \inf(h)$ if and only if $\lim_{x\to a} f(x)$ exists, and

   $$\lim_{x\to a} f(x) = \sup(g) = \inf(h).$$

   Remark: $\inf(h)$ and $\sup(g)$ are called the limit superior and limit inferior of f as it approaches a, respectively. Standard notation for each is $\limsup_{x\to a} f(x)$ and $\liminf_{x\to a} f(x)$.
4. Suppose

   $$G(x) = \inf\{f(y) : y \geq x\},$$
   $$H(x) = \sup\{f(y) : y \geq x\}.$$

   Prove that G is nonincreasing and H is nondecreasing, that $\sup(G) \leq \inf(H)$, and that equality holds iff $\lim_{x\to\infty} f(x)$ exists and is equal to both quantities.

   Remark: $\inf(H)$ and $\sup(G)$ are called the limit superior and limit inferior of f, respectively. Standard notation for each is $\limsup_{x\to\infty} f(x)$ and $\liminf_{x\to\infty} f(x)$.

Problem
Let $f : \mathbb{R} \to \mathbb{R}$ and suppose $M \in \mathbb{R}$ is such that $f(x) \leq M$ for all $x \in \mathbb{R}$. Show that if $\lim_{x\to\infty} f(x)$ exists, then $\lim_{x\to\infty} f(x) \leq M$.

Problem
Suppose $f, g, h : \mathbb{R} \to \mathbb{R}$ are given such that $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R}$. Show that if

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x),$$

then $\lim_{x\to\infty} g(x)$ exists and is equal to the limits of the other two functions.